SELF-CONSISTENT DYNAMICS OF INFLATIONARY PHASE TRANSITIONS [*]

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Abstract

The physics of the inflationary universe requires the study of the out of equilibrium evolution of quantum fields in curved spacetime. We present the evolution for both the geometry and the matter (described by the quantum inflaton field) by means of the non-perturbative large N limit combined with semi-classical gravitational dynamics including the back-reaction of quantum fluctuations self-consistently for a new inflation scenario. We provide a criterion for the validity of the classical approximation and a full analysis of the case in which spinodal quantum fluctuations drive the evolution of the scale factor. Under carefully determined conditions, we show that the full field equations may be well approximated by those of a single composite field which obeys the classical equation of motion in all cases. The de Sitter stage is found to be followed by a matter dominated phase. We compute the spectrum of scalar density perturbations and argue that the spinodal instabilities are responsible for a 'red' spectrum with more power at longer wavelengths. A criterion for the validity of these models is provided and contact with the reconstruction program is established.

I. INTRODUCTION AND MOTIVATION

A. Preliminaries

During the last decade Particle Physics Cosmology matured from the realm of speculation to that of testable predictability. The next generation of satellite and balloon borne experiments will provide an unprecedented test of the fundamental ideas of Early Universe

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Cosmology and will validate or rule out current theories of structure formation and other aspects of the cosmology of the early universe [1,2]. Standard Big Bang cosmology is based on a homogeneous and isotropic expanding universe and is confirmed by observations: i) redshifts of objects far away have been measured and confirm Hubble's law of expansion, ii) the cosmic microwave background radiation (CMBR) has been measured to be an almost perfect blackbody with temperature $T = 2.728 \pm 0.002K$ and is a remnant of the early hot and dense stages of evolution after the Big Bang, iii) light element abundance of $D, {}^3He, {}^4He, {}^7Li$ which is consistent with what is predicted by Nucleosynthesis.

The CMBR gives evidence of homogeneity and isotropy on scales larger than about 100 Mpc, with small temperature fluctuations $\Delta T/T < 10^{-5}$ on angular scales that range from 1° to about 90° [3,4]. This high degree of homogeneity and isotropy on large scales presents one of the important puzzles of standard big bang cosmology, i.e. the horizon problem.

Distances in a spatially flat, homogeneous and isotropic cosmology are measured with the FRW metric

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2, \tag{1.1}$$

with t the comoving or cosmic time and $a(t)$ the scale factor. Physical distances along null geodesics determine the limit of causal correlations. These are given by

$$d_h(t) = a(t) \int^t \frac{dt'}{a(t')}. \tag{1.2}$$

this distance determines the causal horizon, events that are separated by distances larger than $d_h(t)$ cannot be causally correlated by microphysical processes. Another important physical scale is the Hubble radius $d_H(t) = H^{-1}(t) = (\dot{a}(t)/a(t))^{-1}$. In the cases of interest, $d_h(t)$ and $d_H(t)$ are proportional and we will call $d_H(t)$ the horizon indistinguishably. Let us consider a physical distance today $\lambda(t_0)$ over which the CMBR determines homogeneity and isotropy to one part in 10^5. At a time t earlier, the size of this patch is given by $\lambda(t) = \lambda(t_0)a(t)$ (where we have chosen $a(t_0) = 1$). For a matter or radiation dominated universe (most of the life of the Universe), $a(t) \propto t^n$ ($n = 1/2$ for radiation, $n = 2/3$ for matter domination). The horizon size (or Hubble radius) is given by $d_H(t) \propto t$ and the ratio $\lambda(t)/d_H(t)$ increases as we evolve back to an earlier epoch. Thus at some time in the far past, the scale $\lambda(t_0)$ will become larger than the horizon. In particular, if $\lambda(t_0) \approx 100$ Mpc this scale will be larger than the horizon near the time of decoupling of matter and radiation, at red-shifts $z \approx 1100$, and therefore it is extremely difficult to explain how such a region which was causally disconnected at decoupling can be so homogeneous and isotropic. This is the essence of the horizon problem [1,2].

Inflation was proposed over a decade ago to solve long standing problems of Standard Big Bang cosmology such as the homogeneity and horizon problems [5–7]. Inflation corresponds to a stage of exponential expansion of the Universe during which the Hubble radius (horizon) remains constant and physical scales grow exponentially. Thus, in this scenario, microphysical scales smaller than the horizon over which causal processes establish correlations cross the horizon during the inflationary stage. After the inflationary stage and through particle physics processes (reheating), the Universe becomes radiation and matter dominated and the scales that had crossed outside the Hubble radius during the inflationary stage, re-enter

the horizon. Perturbations on these scales then grow under gravitational (Jeans) instability to form the large scale structures that we see today. Thus the inflationary proposal solves the homogeneity and horizon problems by allowing scales to cross the horizon *twice*, from being subhorizon initially becoming superhorizon during inflation, and re-entering the horizon during the period of radiation or matter domination.

Quantum field theory combined with particle physics models provides the framework to implement the inflationary idea. For an inflationary stage the equation of state has to be dominated by a term similar to a cosmological constant leading to a negative pressure. Through the Einstein-Friedmann equations, this yields to an exponential expansion of the scale factor. There are many different inflationary scenarios: 'old', 'new', 'natural', 'hybrid' etc. [1,2], but despite the plethora of different implementations of the original idea, there are some robust features of inflation common to practically all models: i) Inflation predicts a *flat* universe, i.e. the total energy density at all times is the critical energy density, ii) an almost scale invariant spectrum of scalar density perturbations, with amplitudes that are bound by the CMBR inhomogeneities $\Delta T/T$, iii) approximately scale invariant spectrum of gravitational waves (tensor perturbations). The next generation of satellites (MAP/PLANCK) will provide more stringent bounds on temperature inhomogeneities and a firmer determination of the spectrum of scalar and tensor perturbations, and the next generation of gravitational wave detectors (LIGO, VIRGO, LISA) may detect the gravitational waves and their spectrum. Determinations of the spectrum of scalar and tensor perturbations will provide stringent bounds on inflationary models based on particle physics scenarios and probably will validate or rule out specific proposals.

Although the general features of inflation seem robust and universal, the implementation of the inflationary stage as well as the departure from scale invariance of the spectra of scalar and tensor perturbations depend on the concrete model and its dynamics. In this article we focus on the study of the dynamics of specific new inflationary scenarios that lead to an inflationary stage after a phase transition.

B. Dynamics of Phase Transitions

An appealing new inflation scenario envisages a phase transition at GUT scales that provides an inflationary stage [1,2]. The concept is an extension of well understood features of phase transitions in model field theories and has been studied within the context of inflationary cosmology in a fixed background by Linde and Vilenkin [8,9] and Guth and Pi [10]. Consider a scalar field theory (we will neglect fermions, gauge fields and other exotic components in the discussion) with a typical 'Mexican hat' potential that allows for broken symmetry states. If the system is originally at very high temperature, larger than critical, i.e. $T_i > T_c$, the 'effective' potential has a minimum when the expectation value of the scalar field (the order parameter) vanishes. As the temperature cools below critical, the effective potential develops minima away from the origin and the expectation value of the scalar field will roll toward the minima, i.e. the equilibrium values. Consider as a relevant example the following scalar potential

$$V_{eff}(\phi, T(t)) = \frac{\lambda}{4}\,\phi^4 + \frac{1}{2}\,m^2[T(t)]\,\phi^2 + \frac{m^4}{4\lambda}, \qquad (1.3)$$

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$$m^2[T] = m^2 \left[\frac{T^2}{T_c^2} - 1 \right], \tag{1.4}$$

$$T(t) = \frac{T_i}{a(t)} \; ; \;\; T_c \propto \frac{m}{\sqrt{\lambda}}, \tag{1.5}$$

where the time dependence in the temperature results from considering a resummation of the fluctuation contribution to the effective mass in an FRW cosmology at finite temperature [10,11], akin to the hard thermal loop resummation. Assuming that the heat bath arises from particles and radiation at high temperature and that at temperatures larger than the critical the system is in local thermodynamic equilibrium (LTE) with $\phi = 0$, Einstein's equation for the Hubble 'constant' becomes

$$H^2(t) = \frac{8\pi}{3M_{Pl}^2} \left[g\, T^4(t) + \frac{m_R^4}{4\lambda} \right], \tag{1.6}$$

with g being a constant that depends on the particle content of the theory. At the time at which the phase transition takes place, that is when $T(t_{pt}) = T_c$, the Universe is still dominated by radiation, since $T_c^4 \propto m^4/\lambda^2 >> m^4/\lambda$ for weakly coupled theories. Thus we reach the first conclusion: that if a phase transition is driven by the cooling of the expanding Universe, such a phase transition will occur during the radiation dominated era. The inflationary period of exponential (De Sitter) expansion will occur when $T^4(t) << m^4/4\lambda$ when the vacuum energy dominates, which will happen after several e-folds of radiation dominated expansion, once the effective temperature has red-shifted to almost zero.

The small inhomogeneities of the CMBR restrict the scalar self coupling in these models to be $\lambda < 10^{-12}$ [1] (this will also be understood later when we compute the amplitude of scalar perturbations), and assuming LTE may be unjustified in these weakly coupled theories. However, even *if* LTE holds for short wavelength modes of the field theory it is unlikely to hold for the long wavelength modes. The reason for this is the following: at the phase transition when $T(t_{pt}) \approx T_c \approx m/\sqrt{\lambda}$ the Hubble constant is $H \approx m(m/[\lambda M_{Pl}])$. Assuming the De Sitter stage of exponential inflation to occur at a GUT scale, this implies that $m/\lambda^{1/4} \approx 10^{16}$ Gev with the result that at the time of the phase transition $d_H(t_{pt}) << m^{-1}$, i.e. the horizon size is much smaller than the Compton wavelength of the scalar particle. This prevents thermalization of long wavelength modes even when the short wavelength modes may be strongly coupled to the bath and reach LTE. Thus our second conclusion: the phase transition will be strongly supercooled, with the long wavelength modes falling quickly out of LTE, even when LTE prevailed for short wavelength modes. The dynamics must necessarily be studied away from quasi-equilibrium and any approach based on effective potentials will miss important physics.

Thus the phase transition can be described as a sudden quench from the high temperature phase at $T > T_c$ into the low temperature phase with $T << T_c$ on time scales much shorter than the microscopic time scales for thermalization and relaxation [14]. If the initial state was disordered in the sense that the expectation value of the order parameter is zero, then the phase transition occurs with the order parameter sitting at the top of the potential. Because of the symmetry $\phi \to -\phi$ a state with $\phi = \dot{\phi} = 0$ will maintain vanishing value of the order parameter under the dynamics, and the rolling of the field down its potential hill necessary to end the inflationary stage must be understood as a consequence of the quantum fluctuations.

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We have argued previously [13–15] that under these circumstances long wavelength fluctuations will grow almost exponentially in a manner very similar to spinodal decomposition and phase separation in condensed matter systems. The quantum fluctuations as measured by the equal time two-point correlation function of the field $< \Phi^2(\vec{x}, t) >$ must grow [13–15] in such a way that the mean square root fluctuation of the field $\delta\phi = \sqrt{< \Phi^2(\vec{x}, t) >}$ will eventually sample the minima of the potential and reach an equilibrium state. That is to say that at long times, when a quasi-equilibrium state has been achieved, it must be that $< \Phi^2(\vec{x}, t) > \approx m^2/\lambda$. Therefore the quantum fluctuations must become *non-perturbatively* large. Hence we reach a third conclusion: if the expectation value of the scalar field is at the top of the potential hill with vanishingly small velocity during the inflationary epoch, quantum fluctuations that are responsible for the process of phase separation will grow non-perturbatively large. Therefore the dynamics must necessarily be studied within a non-perturbative framework.

Although new inflationary scenarios were previously studied [8,9,12], our work is rather different in that it addresses the description of the dynamics *including* self-consistently and non-perturbatively the non-equilibrium growth of quantum fluctuations and their effect on the dynamics of the scale factor.

C. The puzzling questions:

Having reached this conclusion on the non-perturbative growth of the quantum fluctuations one is presented with unsettling puzzles: typically the scalar field is written as $\Phi(\vec{x}, t) = \phi(t) + \delta\phi(\vec{x}, t)$ where $\phi(t)$ is assumed to be the zero mode or expectation value of the scalar field and $\delta\phi(\vec{x}, t)$ the *small* quantum fluctuations that are ultimately responsible for metric perturbations. Inflation terminates when $\phi(t)$ rolls down the potential hill and reaches the minimum, oscillating about it and eventually decaying into lighter particles leading to the reheating stage.

However, the scenario that is envisaged here is that of a quenched or supercooled phase transition in which $\phi(t) = 0$ throughout the evolution and the fluctuations grow to be very large and to eventually sample the minima of the potential.

Therefore one is led to the questions: a) what is truly rolling down?, b) how does inflation end?, c) how sensitive is the dynamics to the initial value of the expectation value of the scalar field?, d) if the fluctuations grow to become non-perturbatively large will not they provide a large contribution to the energy momentum tensor and modify the FRW dynamics?, e) can one make sense of small fluctuations to calculate density perturbations?.

Even when the initial value of $\phi \neq 0$, an inflationary stage requires that it be sufficiently small for the energy density to be dominated by the vacuum term. Under these circumstances the quantum fluctuations will nevertheless grow, and the dynamics must address not only the rolling of the expectation value, but also the growth of spinodal fluctuations.

The goal of the present work is to answer all of these questions. We will see below that it is precisely the large fluctuations that provide the sensible answers to all of the questions above, thus reconciling the naive picture of a scalar field rolling down the hill. Furthermore, a detailed analysis of the dynamics will allow us to quantify the answer to these questions and, in particular, will allow a profound interpretation of the effective zero mode, its initial conditions, and the small fluctuations that will provide the small density perturbations.

II. THE MODEL AND EQUATIONS OF MOTION

Having recognized the non-perturbative dynamics of the long wavelength fluctuations we need to study the dynamics within a non-perturbative framework. We require that such framework be: i) renormalizable, ii) covariant energy conserving, iii) numerically implementable. There are very few schemes that fulfill all of these criteria: the Hartree and the large N approximation [9,13,14]. Whereas the Hartree approximation is basically a Gaussian variational approximation [16,17] that in general cannot be consistently improved upon, the large N approximation can be consistently implemented beyond leading order [18,19] and in our case it has the added bonus of providing many light fields (associated with Goldstone modes) that will permit the study the effects of other fields which are lighter than the inflaton on the dynamics. Thus we will study the inflationary dynamics of a quenched phase transition within the framework of the large N limit of a scalar theory in the vector representation of $O(N)$.

We assume that the universe is spatially flat with a metric given by eq. (1.1). The matter action and Lagrangian density are given by

$$S_m = \int d^4x \, \mathcal{L}_m = \int d^4x \, a^3(t) \left[\frac{1}{2}\dot{\vec{\Phi}}^2(x) - \frac{1}{2}\frac{(\vec{\nabla}\vec{\Phi}(x))^2}{a^2(t)} - V(\vec{\Phi}(x)) \right] \tag{2.1}$$

$$V(\vec{\Phi}) = \frac{\lambda}{8N}\left(\vec{\Phi}^2 + \frac{2NM^2}{\lambda}\right)^2 \quad ; \quad M^2 = -m^2 + \xi\,\mathcal{R}, \tag{2.2}$$

$$\mathcal{R} = 6\left(\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)}\right), \tag{2.3}$$

where we have included the coupling of $\Phi(x)$ to the scalar curvature $\mathcal{R}(t)$ since it will arise as a consequence of renormalization [11].

The gravitational sector includes the usual Einstein term in addition to a higher order curvature term and a cosmological constant term which are necessary to renormalize the theory. The Lagrangian density for the gravitational sector is therefore:

$$\mathcal{L}_g = a^3(t)\left[\frac{\mathcal{R}(t)}{16\pi G} + \frac{\alpha}{2}\mathcal{R}^2(t) - K\right].$$

with K being the cosmological constant. In principle, we also need to include the terms $R^{\mu\nu}R_{\mu\nu}$ and $R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ as they are also terms of fourth order in derivatives of the metric (fourth adiabatic order), but the variations resulting from these terms turn out not to be independent of that of $\mathcal{R}^2$ in the flat FRW cosmology we are considering.

The variation of the action (2.1) with respect to the metric $g_{\mu\nu}$ gives us Einstein's equation

$$\frac{G_{\mu\nu}}{8\pi G} + \alpha H_{\mu\nu} + Kg_{\mu\nu} = -T_{\mu\nu}, \tag{2.4}$$

where $G_{\mu\nu}$ is the Einstein tensor given by the variation of $\sqrt{-g}\mathcal{R}$, $H_{\mu\nu}$ is the higher order curvature term given by the variation of $\sqrt{-g}\mathcal{R}^2$, and $T_{\mu\nu}$ is the contribution from the

matter Lagrangian. With the metric (1.1), the various components of the curvature tensors in terms of the scale factor are:

$$G^0_0 = -3(\dot{a}/a)^2, \tag{2.5}$$

$$G^\mu_\mu = -\mathcal{R} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \tag{2.6}$$

$$H^0_0 = -6\left(\frac{\dot{a}}{a}\dot{\mathcal{R}} + \frac{\dot{a}^2}{a^2}\mathcal{R} - \frac{1}{12}\mathcal{R}^2\right), \tag{2.7}$$

$$H^\mu_\mu = -6\left(\ddot{\mathcal{R}} + 3\frac{\dot{a}}{a}\dot{\mathcal{R}}\right). \tag{2.8}$$

Eventually, when we have fully renormalized the theory, we will set $\alpha_R = 0$ and keep as our only contribution to K_R a piece related to the matter fields which we shall incorporate into $T_{\mu\nu}$.

A. The Large N Approximation

To obtain the proper large N limit, the vector field is written as

$$\vec{\Phi}(\vec{x}, t) = (\sigma(\vec{x}, t), \vec{\pi}(\vec{x}, t)),$$

with $\vec{\pi}$ an $N - 1$-plet, and we write

$$\sigma(\vec{x}, t) = \sqrt{N}\phi(t) + \chi(\vec{x}, t) \ ; \ \langle\sigma(\vec{x}, t)\rangle = \sqrt{N}\phi(t) \ ; \ \langle\chi(\vec{x}, t)\rangle = 0. \tag{2.9}$$

To implement the large N limit in a consistent manner, one may introduce an auxiliary field as in [19]. However, the leading order contribution can be obtained equivalently by invoking the factorization [14,20]:

$$\chi^4 \to 6\langle\chi^2\rangle\chi^2 + \text{ constant }, \tag{2.10}$$

$$\chi^3 \to 3\langle\chi^2\rangle\chi, \tag{2.11}$$

$$(\vec{\pi} \cdot \vec{\pi})^2 \to 2\langle\vec{\pi}^2\rangle\vec{\pi}^2 - \langle\vec{\pi}^2\rangle^2 + \mathcal{O}(1/N), \tag{2.12}$$

$$\vec{\pi}^2\chi^2 \to \langle\vec{\pi}^2\rangle\chi^2 + \vec{\pi}^2\langle\chi^2\rangle, \tag{2.13}$$

$$\vec{\pi}^2\chi \to \langle\vec{\pi}^2\rangle\chi. \tag{2.14}$$

To obtain a large N limit, we define [14,20]

$$\vec{\pi}(\vec{x}, t) = \psi(\vec{x}, t)\overbrace{(1, 1, \cdots, 1)}^{N-1}, \tag{2.15}$$

where the large N limit is implemented by the requirement that

$$\langle\psi^2\rangle \approx \mathcal{O}(1) \ , \ \langle\chi^2\rangle \approx \mathcal{O}(1) \ , \ \phi \approx \mathcal{O}(1). \tag{2.16}$$

The leading contribution is obtained by neglecting the $\mathcal{O}(1/N)$ terms in the formal limit. The resulting Lagrangian density is quadratic, with linear terms in χ and $\vec{\pi}$. The equations of

motion are obtained by imposing the tadpole conditions $< \chi(\vec{x}, t) >= 0$ and $< \vec{\pi}(\vec{x}, t) >= 0$ which in this case are tantamount to requiring that the linear terms in χ and $\vec{\pi}$ in the Lagrangian density vanish. Since the action is quadratic, the quantum fields can be expanded in terms of creation and annihilation operators and mode functions that obey the Heisenberg equations of motion

$$\vec{\pi}(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3} \left[\vec{a}_k f_k(t) e^{i\vec{k}\cdot\vec{x}} + \vec{a}_k^\dagger f_k^*(t) e^{-i\vec{k}\cdot\vec{x}} \right]. \tag{2.17}$$

The tadpole condition leads to the following equations of motion [14,20]:

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + \mathcal{M}^2(t)\phi(t) = 0, \tag{2.18}$$

with the mode functions

$$\left[\frac{d^2}{dt^2} + 3H(t)\frac{d}{dt} + \frac{k^2}{a^2(t)} + \mathcal{M}^2(t) \right] f_k(t) = 0, \tag{2.19}$$

where

$$\mathcal{M}^2(t) = -m^2 + \xi\mathcal{R} + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\langle\psi^2(t)\rangle. \tag{2.20}$$

In this leading order in $1/N$ the theory becomes Gaussian, but with the self-consistency condition

$$\langle\psi^2(t)\rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{|f_k(t)|^2}{2}. \tag{2.21}$$

The initial conditions on the modes $f_k(t)$ must now be determined. At this stage it proves illuminating to pass to conformal time variables in terms of the conformally rescaled fields (see [20] for a discussion) in which the mode functions obey an equation which is very similar to that of harmonic oscillators with time dependent frequencies in Minkowski space-time. It has been realized that different initial conditions on the mode functions lead to different renormalization counterterms [20]; in particular imposing initial conditions in comoving time leads to counterterms that depend on these initial conditions. Thus we chose to impose initial conditions in conformal time in terms of the conformally rescaled mode functions leading to the following initial conditions in comoving time:

$$f_k(t_0) = \frac{1}{\sqrt{W_k}}, \quad \dot{f}_k(t_0) = \left[-\frac{\dot{a}(t_0)}{a(t_0)} - iW_k \right] f_k(t_0), \tag{2.22}$$

with

$$W_k^2 \equiv k^2 + \mathcal{M}^2(t_0) - \frac{\mathcal{R}(t_0)}{6}. \tag{2.23}$$

At this point we recognize that when $\mathcal{M}^2(t_0) - \mathcal{R}(t_0)/6 < 0$ the above initial condition must be modified to avoid imaginary frequencies, which are the signal of instabilities for long wavelength modes. Thus we *define* the initial frequencies that determine the initial conditions (2.22) as

$$W_k^2 \equiv k^2 + \left| M^2(t_0) - \frac{\mathcal{R}(t_0)}{6} \right| \quad \text{for } k^2 < \left| M^2(t_0) - \frac{\mathcal{R}(t_0)}{6} \right|, \tag{2.24}$$

$$W_k^2 \equiv k^2 + M^2(t_0) - \frac{\mathcal{R}(t_0)}{6} \quad \text{for } k^2 \geq \left| M^2(t_0) - \frac{\mathcal{R}(t_0)}{6} \right|. \tag{2.25}$$

In the large N limit we find the energy density and pressure density to be given by,

$$\frac{\varepsilon}{N} = \frac{1}{2}\dot{\phi}^2(t) + \frac{\lambda}{8}\left(\phi^2(t) + \frac{2M^2}{\lambda} \right)^2$$

$$+ \frac{1}{2} \int \frac{d^3k}{2(2\pi)^3} \left[|\dot{f}_k(t)|^2 + \omega_k^2(t)|f_k(t)|^2 \right] - \frac{\lambda}{8}\langle \psi^2(t) \rangle^2, \tag{2.26}$$

$$\frac{p + \varepsilon}{N} = \dot{\phi}^2(t) + \int \frac{d^3k}{2(2\pi)^3} \left[|\dot{f}_k(t)|^2 + \frac{k^2}{3a^2(t)}|f_k(t)|^2 \right]. \tag{2.27}$$

It is straightforward to show that the bare energy is covariantly conserved by using the equations of motion for the zero mode and the mode functions.

III. RENORMALIZATION

Renormalization is a very subtle but important issue in gravitational backgrounds [21]. The fluctuation contribution $\langle \psi^2(\vec{x}, t) \rangle$, the energy, and the pressure all need to be renormalized. The renormalization aspects in curved space times have been discussed at length in the literature [21] and has been extended to the large N self-consistent approximations for the non-equilibrium backreaction problem in [19,20]. More recently a consistent and covariant regularization scheme that can be implemented numerically has been provided (see J. Baacke's contribution to these proceedings and [22,23]).

In terms of the effective mass term for the large N limit given by (2.20) and defining the quantity

$$B(t) \equiv a^2(t) \left(\mathcal{M}^2(t) - \mathcal{R}/6 \right), \tag{3.1}$$

$$\mathcal{M}^2(t) = -m_B^2 + \xi_B \mathcal{R}(t) + \frac{\lambda_B}{2}\phi^2(t) + \frac{\lambda_B}{2}\langle \psi^2(t) \rangle_B, \tag{3.2}$$

where the subscript B stands for bare quantities, we find the following large k behavior for the case of an *arbitrary* scale factor $a(t)$ (with $a(0) = 1$):

$$|f_k(t)|^2 = \frac{1}{ka^2(t)} + \frac{1}{2k^3a^2(t)}[-B(t)]$$

$$+ \frac{1}{8a(t)^2\,k^5}\left\{ B(t)[3B(t)] + a(t)\frac{d}{dt}\left[a(t)\dot{B}(t) \right] \right\} + \mathcal{O}(1/k^7) \tag{3.3}$$

$$|\dot{f}_k(t)|^2 = \frac{k}{a^4(t)} + \frac{1}{ka^2(t)}\left[H^2(t) + \frac{1}{2}\left(\mathcal{M}^2(t) - \mathcal{R}/6 \right) \right]$$

$$+ \frac{1}{8a(t)^4\,k^3}\left\{ -B(t)^2 - a(t)^2\ddot{B}(t) + 3a(t)\dot{a}(t)\dot{B}(t) - 4\dot{a}^2(t)B(t) \right\} + \mathcal{O}(1/k^5). \tag{3.4}$$

Although the divergences can be dealt with by dimensional regularization, this procedure is not well suited to numerical analysis (see however ref. [22]). We will make our subtractions using an ultraviolet cutoff constant in *physical coordinates*. This guarantees that the counterterms will be time independent. The renormalization then proceeds much in the same manner as in reference [11]; the quadratic divergences renormalize the mass and the logarithmic terms renormalize coupling constant and the coupling to the Ricci scalar. The renormalization conditions on the mass, coupling to the Ricci scalar and coupling constant are obtained from the requirement that the frequencies that appear in the mode equations are finite [11], i.e:

$$-m_B^2 + \xi_B \mathcal{R}(t) + \frac{\lambda_B}{2}\phi^2(t) + \frac{\lambda_B}{2}\langle\psi^2(t)\rangle_B = -m_R^2 + \xi_R \mathcal{R}(t) + \frac{\lambda_R}{2}\phi^2(t) + \frac{\lambda_R}{2}\langle\psi^2(t)\rangle_R \quad (3.5)$$

Finally we arrive at the following set of renormalizations [20]:

$$\frac{1}{8\pi G_R} = \frac{1}{8\pi G_B} - 2\left(\xi_R - \frac{1}{6}\right)\frac{\Lambda^2}{16\pi^2} - 2\left(\xi_R - \frac{1}{6}\right)m_R^2\frac{\ln(\Lambda/\kappa)}{16\pi^2}, \quad (3.6)$$

$$\alpha_R = \alpha_B - \left(\xi_R - \frac{1}{6}\right)^2\frac{\ln(\Lambda/\kappa)}{16\pi^2}, \quad (3.7)$$

$$K_R = K_B - \frac{\Lambda^4}{16\pi^2} - m_R^2\frac{\Lambda^2}{16\pi^2} + \frac{m_R^4}{2}\frac{\ln(\Lambda/\kappa)}{16\pi^2}, \quad (3.8)$$

$$m_R^2 = m_B^2 + \lambda_R\frac{\Lambda^2}{16\pi^2} - \lambda_R m_R^2\frac{\ln(\Lambda/\kappa)}{16\pi^2}, \quad (3.9)$$

$$\xi_R = \xi_B - \lambda_R\left(\xi_R - \frac{1}{6}\right)\frac{\ln(\Lambda/\kappa)}{16\pi^2}, \quad (3.10)$$

$$\lambda_R = \lambda_B - \lambda_R\frac{\ln(\Lambda/\kappa)}{16\pi^2}, \quad (3.11)$$

$$\langle\psi^2(t)\rangle = \int\frac{d^3k}{2(2\pi)^3}\left\{|f_k(t)|^2 - \left[\frac{1}{ka^2(t)} - \frac{\Theta(|\vec{k}| - \kappa)}{2k^3a^2(t)}\left[\mathcal{M}^2(t) - \mathcal{R}/6\right]\right]\right\}. \quad (3.12)$$

Here, κ is the renormalization point. As expected, the logarithmic terms are consistent with the renormalizations found using dimensional regularization [22,23]. Again, we set $\alpha_R = 0$ and choose the renormalized cosmological constant such that the vacuum energy is zero in the true vacuum. We emphasize that while the regulator we have chosen does not respect the covariance of the theory, the renormalized energy momentum tensor defined in this way nevertheless retains the property of covariant conservation in the limit when the cutoff is taken to infinity. In what follows, we drop the subscripts from the renormalized parameters.

The logarithmic subtractions can be neglected because of the coupling $\lambda \leq 10^{-12}$. Using the Planck scale as the cutoff and the inflaton mass m_R as a renormalization point, these terms are of order $\lambda\ln[M_{pl}/m_R] \leq 10^{-10}$, for $m \geq 10^9$ GeV . An equivalent statement is that for these values of the coupling and inflaton masses, the Landau pole is well beyond the physical cutoff M_{pl}. Our relative error in the numerical analysis is of order 10^{-8}, therefore our numerical study is insensitive to the logarithmic corrections. Though these corrections are fundamentally important, numerically they can be neglected. Therefore, in what follows, we will neglect logarithmic renormalization and subtract only quartic and quadratic divergences in the energy and pressure, and quadratic divergences in the fluctuation contribution.

IV. RENORMALIZED EQUATIONS OF MOTION FOR DYNAMICAL EVOLUTION

It is convenient to introduce the following dimensionless quantities and definitions,

$$\tau = m_R t \quad ; \quad h = \frac{H}{m_R} \quad ; \quad q = \frac{k}{m_R} \qquad \omega_q = \frac{W_k}{m_R} \quad ; \quad g = \frac{\lambda_R}{8\pi^2} \,, \tag{4.1}$$

$$\eta^2(\tau) = \frac{\lambda_R}{2m_R^2} \, \phi^2(t) \quad ; \quad g\Sigma(\tau) = \frac{\lambda}{2m_R^2} \, \langle \psi^2(t) \rangle_R \quad ; \quad f_q(\tau) \equiv \sqrt{m_R} \, f_k(t) \,. \tag{4.2}$$

Choosing $\xi_R = 0$ (minimal coupling) and the renormalization point $\kappa = |m_R|$ and setting $a(0) = 1$, the equations of motion become:

$$\left[\frac{d^2}{d\tau^2} + 3h\frac{d}{d\tau} - 1 + \eta^2(\tau) + g\Sigma(\tau) \right] \eta(\tau) = 0, \tag{4.3}$$

$$\left[\frac{d^2}{d\tau^2} + 3h\frac{d}{d\tau} + \frac{q^2}{a^2(\tau)} - 1 + \eta^2 + g\Sigma(\tau) \right] f_q(\tau) = 0,$$

$$f_q(0) = \frac{1}{\sqrt{\omega_q}} \quad ; \quad \dot{f}_q(0) = [-h(0) - i\omega_q] \, f_q(0),$$

$$\omega_q = \left[q^2 - 1 + \eta^2(0) - \frac{\mathcal{R}(0)}{6m_R^2} + g\Sigma(0) \right]^{\frac{1}{2}} \quad \text{for} \quad q^2 > -1 + \eta^2(0) - \frac{\mathcal{R}(0)}{6m_R^2} + g\Sigma(0),$$

$$\omega_q = \left[q^2 + 1 - \eta^2(0) + \frac{\mathcal{R}(0)}{6m_R^2} - g\Sigma(0) \right]^{\frac{1}{2}} \quad \text{for} \quad q^2 < -1 + \eta^2(0) - \frac{\mathcal{R}(0)}{6m_R^2} + g\Sigma(0). \tag{4.4}$$

The initial conditions for $\eta(\tau)$ will be specified later. An important point to notice is that the equation of motion for the $q = 0$ mode coincides with that of the zero mode (4.3). Furthermore, for $\eta(\tau \to \infty) \neq 0$, a stationary (equilibrium) solution of the eq. (4.3) is obtained when the sum rule [13,14,20]

$$-1 + \eta^2(\infty) + g\Sigma(\infty) = 0 \tag{4.5}$$

is fulfilled. This sum rule is nothing but a consequence of Goldstone's theorem and is a result of the fact that the large N approximation satisfies the Ward identities associated with the $O(N)$ symmetry, since the term $-1 + \eta^2 + g\Sigma$ is seen to be the effective mass of the modes transverse to the symmetry breaking direction, i.e. the Goldstone modes in the broken symmetry phase.

In terms of the zero mode $\eta(\tau)$ and the quantum mode function given by eq.(4.4) we find that the Friedmann equation for the dynamics of the scale factor in dimensionless variables is given by

$$h^2(\tau) = 4h_0^2 \epsilon_R(\tau) \quad ; \quad h_0^2 = \frac{4\pi N m_R^2}{3M_{Pl}^2 \lambda_R} \tag{4.6}$$

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and the renormalized energy and pressure are given by:

$$\epsilon_R(\tau) = \frac{1}{2}\dot{\eta}^2 + \frac{1}{4}\left(-1+\eta^2+g\Sigma\right)^2 +$$

$$\frac{g}{2}\int q^2 dq \left[\left(|\dot{f}_q|^2 - \mathcal{S}^{(1)}(q,\tau)\right) + \frac{q^2}{a^2}\left(|f_q|^2 - \mathcal{S}^{(2)}(q,\tau)\right)\right] , \tag{4.7}$$

$$(p+\varepsilon)_R = \frac{2Nm_R^4}{\lambda_R}\left\{\frac{1}{2}\dot{\eta}^2 + g\int q^2 dq \left[|\dot{f}_q|^2 - \mathcal{S}^1(q,\tau) + \frac{q^2}{3a^2}\left(|f_q|^2 - \mathcal{S}^{(2)}(q,\tau)\right)\right]\right\}, \tag{4.8}$$

where the subtractions $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are given by the right hand sides of eqns.(3.4) and (3.3) respectively.

We want to emphasize the following aspects of the set of equations (4.3,4.4,4.7): they determine the full dynamics of the matter plus classical gravity including backreaction effects both on the metric as well as in the dynamics of the fields, they are fully renormalized, maintain covariant conservation and in principle they can be consistently improved in the $1/N$ expansion.

In order to provide the full solution we now must provide the values of $\eta(0)$, $\dot{\eta}(0)$, and h_0. Assuming that the inflationary epoch is associated with a phase transition at the GUT scale, this requires that $Nm_R^4/\lambda_R \approx (10^{15} \text{ Gev })^4$ and assuming the bound on the scalar self-coupling $\lambda_R \approx 10^{-12} - 10^{-14}$ (this will be seen later to be a compatible requirement), we find that $h_0 \approx N^{1/4}$ which we will take to be reasonably given by $h_0 \approx 1 - 10$ (for example in popular GUT's $N \approx 20$ depending on particular representations).

We will begin by studying the case of most interest from the point of view of describing the phase transition: $\eta(0) = 0$ and $\dot{\eta}(0) = 0$, which are the initial conditions that led to our puzzling questions. With these initial conditions, the evolution equation for the zero mode eq. (4.3) determines that $\eta(\tau) = 0$ by symmetry.

A. Early time dynamics:

Before engaging in a full numerical study, it proves illuminating to obtain an estimate of the relevant time scales and an intuitive idea of the main features of the dynamics. Because the coupling is so weak and after renormalization the contribution from the quantum fluctuations to the equations of motion is finite, we can obtain an estimate of the early time dynamics by neglecting the backreaction terms in the equations for the mode functions (4.4) and the Hubble constant (4.6). Setting $\eta = 0$ and $g\Sigma \approx 0$ in eq. (4.4) and also setting to zero the terms proportional to g in eq.(4.7), the evolution equations for the mode functions are those for an inverted oscillator in De Sitter space-time, which have been studied by Guth and Pi [10]. One obtains the approximate solution

$$h(t) \approx h_0,$$

$$f_q(t) \approx e^{-3h_0\tau/2}\left[A_q \, J_\nu\left(\frac{q}{h_0}e^{-h_0\tau}\right) + B_q \, J_{-\nu}\left(\frac{q}{h_0}e^{-h_0\tau}\right)\right],$$

$$\nu = \sqrt{\frac{9}{4} + \frac{1}{h_0^2}}, \tag{4.9}$$

where $J_{\pm\nu}(z)$ are Bessel functions, and A_q and B_q are determined by the initial conditions on the mode functions.

When the physical wavevectors cross the horizon, i.e. when $qe^{-h_0\tau}/h_0 \ll 1$ we find that the mode functions factorize:

$$f_q(\tau) \approx \frac{B_q}{\Gamma(1-\nu)} \left(\frac{2h_0}{q}\right)^\nu e^{(\nu-3/2)h_0\tau}. \tag{4.10}$$

This result reveals a very important feature: $\nu > 3/2$; because of the negative squared mass term in the matter Lagrangian leading to symmetry breaking, we see that all of the mode functions *grow exponentially* after horizon crossing (for positive squared mass they would *decrease exponentially* after horizon crossing). This exponential growth is a consequence of the spinodal instabilities which is modified in De Sitter space-time but is a hallmark of the process of phase separation that occurs to complete the phase transition. We note, in addition that the time dependence is exactly given by that of the $q = 0$ mode, i.e. the zero mode, which is a consequence of the redshifting of the wavevectors and the fact that after horizon crossing the contribution of the term $q^2/a^2(\tau)$ in the equations of motion become negligible. Then we clearly see that the quantum fluctuations grow exponentially and they will begin to be of the order of the tree level terms in the equations of motion when $g\Sigma(\tau) \approx 1$. At large times $\Sigma(\tau) \approx \mathcal{F}^2(h_0)(h_0/2\pi)^2 e^{(2\nu-3)h_0\tau}$, with $\mathcal{F}(h_0)$ a finite constant that depends on the initial conditions and is found numerically to be of $\mathcal{O}(1)$.

In terms of the initial dimensionfull variables, this condition translates to $< \psi^2(\vec{x},t) >_R \approx m_R^2/\lambda_R$, i.e. the quantum fluctuations sample the minima of the (renormalized) tree level potential. We find that the time at which the contribution of the quantum fluctuations becomes of the same order as the tree level terms is estimated to be [14]

$$\tau_s \approx \frac{1}{(2\nu-3)h_0} \ln\left(\frac{32\pi^4}{\lambda h_0^2 \mathcal{F}^2(h_0)}\right) = \frac{3}{2}h_0 \ln\left(\frac{32\pi^4}{\lambda h_0^2 \mathcal{F}^2(h_0)}\right) + \mathcal{O}(1/h_0). \tag{4.11}$$

At this time, the contribution of the quantum fluctuations makes the back reaction very important and, as will be seen numerically, this translates into the fact that τ_s also determines the end of the De Sitter era and the end of inflation. The total number of e-folds during the stage of exponential expansion of the scale factor (constant h_0) is given by

$$N_e \approx \frac{1}{(2\nu-3)} \ln\left(\frac{32\pi^4}{\lambda h_0^2 \mathcal{F}^2(h_0)}\right) = \frac{3}{2}h_0^2 \ln\left(\frac{32\pi^4}{\lambda h_0^2 \mathcal{F}^2(h_0)}\right) + \mathcal{O}(1/h_0) \tag{4.12}$$

For large h_0 we see that the number of efolds scales as h_0^2 as well as with the logarithm of the inverse coupling. These results (4.10,4.11,4.12) will be confirmed numerically below and will be of paramount importance for the interpretation of the main consequences of the dynamical evolution.

B. $\eta(0) \neq 0$: classical or quantum behavior?

Above we have analyzed the situation when $\eta(0) = 0$ (or in dimensionfull variables $\phi(0) = 0$). The typical analysis of inflaton dynamics in the literature involves the *classical*

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evolution of $\phi(t)$ with an initial condition in which $\phi(0)$ is very close to zero (i.e. the top of the potential hill) in the 'slow-roll' regime, for which $\ddot{\phi} \ll 3H\dot{\phi}$. Thus, it is important to quantify the initial conditions on ϕ for which the dynamics will be determined by the classical evolution of ϕ and those for which the quantum fluctuations dominate the dynamics. We can provide a criterion to separate classical from quantum dynamics by analyzing the relevant time scales, estimated by neglecting non-linearities and backreaction effects. Considering the linear evolution of the zero mode in terms of dimensionless variables, and considering $\eta(0) \neq 0$ ($\dot{\eta}(0) \neq 0$ simply corresponds to a shift in origin of time), we find

$$\eta(\tau) \approx \eta(0)e^{(\nu - \frac{3}{2})h_0\tau}. \tag{4.13}$$

The non-linearities will become important and eventually terminate inflation when $\eta(\tau) \approx 1$. This corresponds to a time

$$\tau_c \approx \frac{\ln(1/\eta(0))}{(\nu - \frac{3}{2})h_0}. \tag{4.14}$$

If τ_c is much smaller than the spinodal time τ_s given by eq.(4.11) then the *classical* evolution of the zero mode will dominate the dynamics and the quantum fluctuations will not become very large, although they will still undergo spinodal growth. On the other hand, if $\tau_c \gg \tau_s$ the quantum fluctuations will grow to be very large well before the zero mode reaches the non-linear regime. In this case the dynamics will be determined completely by the quantum fluctuations. Then the criterion for the classical or quantum dynamics is given by

$$\eta(0) \gg \sqrt{\lambda}h_0 \implies \text{classical dynamics}$$
$$\eta(0) \ll \sqrt{\lambda}h_0 \implies \text{quantum dynamics} \tag{4.15}$$

or in terms of dimensionfull variables $\phi(0) \gg H_0$ leads to *classical dynamics* and $\phi(0) \ll H_0$ leads to *quantum dynamics*.

However, even when the classical evolution of the zero mode dominates the dynamics, the quantum fluctuations grow exponentially after horizon crossing unless the value of ϕ is very close to the minimum of the tree level potential. In the large N approximation the spinodal line, that is the values of ϕ for which there are spinodal instabilities, reaches all the way to the minimum of the tree level potential as can be seen from the equations of motion for the mode functions. Therefore even in the classical case one must understand how to deal with quantum fluctuations that grow and become large after horizon crossing.

C. Numerics

The time evolution is carried out by means of a fourth order Runge-Kutta routine with adaptive stepsizing while the momentum integrals are carried out using an 11-point Newton-Cotes integrator. The relative errors in both the differential equation and the integration are of order 10^{-8}. We find that the energy is covariantly conserved throughout the evolution to better than a part in a thousand. Figures (1-3) show $g\Sigma(\tau)$ vs τ, $h(\tau)$ vs. τ and $\ln(|f_q(\tau)|^2)$ vs τ for several values of q with larger $q's$ corresponding to successively lower curves. Figures

(4,5) show $p(\tau)/\varepsilon(\tau)$ and the horizon size $h^{-1}(\tau)$ for $\lambda = 10^{-12}$; $\eta(0) = 0$; $\dot{\eta}(0) = 0$ and we have chosen the representative value $h_0 = 1.5$.

Figures 1 and 2 show clearly that when the contribution of the quantum fluctuations $g\Sigma(\tau)$ becomes of order 1 inflation basically ends, and the time scale for $g\Sigma$ to reach $\mathcal{O}(1)$ is very well described by the estimate (4.11). From figure 1 we see that this happens for $\tau \approx 75$, leading to a number of e-folds $N_e \approx 110$ which is correctly estimated by (4.11, 4.12).

Figure 3 shows clearly the factorization of the modes after they cross the horizon given by eq. (4.10). The slopes of all the curves after they become straight lines in figure 3 is given exactly by $(2\nu - 3)$, whereas the intercept depends on the initial condition on the mode function and the larger the value of q the smaller the intercept because the amplitude of the mode function is smaller initially. Although the intercept depends on the initial conditions on the long-wavelength modes, the slope is independent of the value of q and is the same as what would be obtained in the linear approximation for the *square* of the zero mode at times long enough that the decaying solution can be neglected but short enough that the effect of the non-linearities is very small. Notice from the figure that when inflation ends and the non-linearities become important all of the modes basically saturate. This is also what one would expect from the solution of the zero mode: exponential growth in early-intermediate times (neglecting the decaying solution), with a growth exponent given by $(\nu - 3/2)$ and an asymptotic behavior of small oscillations around the equilibrium position, which for the zero mode is $\eta = 1$, but for the $q \neq 0$ modes depends on the initial conditions. All of the mode functions have this behavior once they cross the horizon. We have also studied the phases of the mode functions and we found that they 'freeze' after horizon crossing. This is natural since both the real and imaginary parts of f_q obey the same equation but with different boundary conditions. After the physical wavelength crosses the horizon, the dynamics is insensitive to the value of q for real and imaginary parts and the phases become independent of time. Again, this is a consequence of factorization.

The growth of the quantum fluctuations is sufficient to end inflation at a time given by τ_s in eq. (4.11). Furthermore figure 4 shows that during the inflationary epoch $p(\tau)/\varepsilon(\tau) \approx -1$ and the end of inflation is rather sharp at τ_s with $p(\tau)/\varepsilon(\tau)$ oscillating between ± 1 with zero average over the cycles, resulting in matter domination. Figure 5 shows this feature very clearly; $h(\tau)$ is constant during the De Sitter epoch and becomes matter dominated after the end of inflation with $h^{-1}(\tau) \approx 3(\tau - \tau_f)/2$, with τ_f the time at the end of inflation. There are small oscillations around this value because both $p(\tau)$ and $\varepsilon(\tau)$ oscillate. These oscillations are a result of small oscillations of the mode functions after they saturate, and are also a feature of the solution for a zero mode.

D. Zero Mode Assembly

This remarkable feature of factorization of the mode functions after horizon crossing can be elegantly summarized as

$$f_k(t)|_{k_{ph}(t) \ll H} = g(q, h) f_0(\tau), \tag{4.16}$$

with $f_0(\tau)$ a real function that obeys the zero mode equation (4.3), $k_{ph}(t) = ke^{-Ht}$ the physical momentum, and $g(q, h)$ a complex constant. Since the factor $g(q, h)$ depends solely

on the initial conditions on the mode functions, it turns out that for two mode functions corresponding to momenta k_1, k_2 that have crossed the horizon at times $t_1 > t_2$, the ratio of the two mode functions at time $t_s > t > t_1 > t_2$ is $f_{k_1}(t)/f_{k_2}(t) \propto e^{(\nu-\frac{3}{2})h(\tau_1-\tau_2)} > 1$. Then if we consider the contribution of these modes to the *renormalized* quantum fluctuations a long time after the beginning of inflation (so as to neglect the decaying solutions), we find that $g\Sigma(\tau) \approx \mathcal{C}e^{(2\nu-3)h\tau} +$ small , where 'small' stands for the contribution of mode functions associated with momenta that have not yet crossed the horizon at time τ, which give a perturbatively small (of order λ) contribution. Then it is clear that after several e-folds from the beginning of inflation, we can define an 'effective zero mode' as

$$\eta^2_{eff}(\tau) \equiv g\Sigma(\tau), \text{ or in dimensionfull variables, } \phi_{eff}(t) \equiv \left[\langle\psi^2(\vec{x},t)\rangle_R\right]^{\frac{1}{2}}. \qquad (4.17)$$

Although this identification seems natural, we emphasize that it is by no means a trivial or ad-hoc statement. There are several important features that allow an *unambiguous* identification: i) $[\langle\psi^2(\vec{x},t)\rangle_R]$ is a fully renormalized operator and hence finite, ii) because of the factorization of the superhorizon modes that enter in the evaluation of $[\langle\psi^2(\vec{x},t)\rangle_R]$, $\phi_{eff}(t)$ (4.17) *obeys the equation of motion for the zero mode*, iii) this identification is valid several e-folds after the beginning of inflation, after the transient decaying solutions have died away and the integral in $\langle\psi^2(\vec{x},t)\rangle$ is dominated by the modes with wavevector k that have crossed the horizon at $t(k) << t$. Numerically we see that this identification holds throughout the dynamics but for a very few e-folds at the beginning of inflation. This factorization determines at once the initial conditions of the effective zero mode that can be extracted numerically: after the first few efolds and long before the end of inflation we find

$$\phi_{eff}(t) \equiv \phi_{eff}(0)e^{(\nu-\frac{3}{2})Ht} \;\; ; \;\; \phi_{eff}(0) \equiv \frac{H}{2\pi}\mathcal{F}(H/m), \qquad (4.18)$$

where we parametrized $\phi_{eff}(0) \equiv \frac{H}{2\pi}\mathcal{F}(H/m)$ to make contact with the literature. We find numerically that $\mathcal{F}(H/m) \approx \mathcal{O}(1)$ for a large range of $0.1 \leq H/m \leq 50$ and depends on the initial conditions of the long wavelength modes.

The factorization of the superhorizon modes (that dominate the integral) implies that

$$g \int q^2 dq |f_q^2(\tau)| \to C_0^2 \, |f_0^2(\tau)|, \qquad (4.19)$$

$$g \int q^2 dq |\dot{f}_q^2(\tau)| \to C_0^2 \, |\dot{f}_0^2(\tau)|, \qquad (4.20)$$

$$g \int \frac{q^4}{a^2(\tau)} dq |f_q^2(\tau)| \to C_1^2 \, \frac{|f_0^2(\tau)|}{a^2(\tau)}. \qquad (4.21)$$

We also find numerically that even when $\eta(0) \neq 0$ this factorization phenomenon is very robust, and after a few e-folds from the beginning of inflation the dynamics is completely determined by the effective zero mode

$$\eta_{eff}(\tau) \equiv \sqrt{\eta^2(\tau) + g\Sigma(\tau)}. \qquad (4.22)$$

We have checked numerically that the dynamics of the scale factor and equation of state obtained from the full quantum problem is exactly equivalent to that obtained from the

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classical problem in terms of $\eta_{eff}(\tau)$. We have also checked numerically that the estimate for the classical to quantum crossover given by eq.(4.15) is quantitatively correct. Thus in the classical case in which $\eta(0) \gg \sqrt{\lambda}$ we find that $\eta_{eff}(\tau) = \eta(\tau)$ whereas in the opposite, quantum case $\eta_{eff}(\tau) = \sqrt{g\Sigma(\tau)}$. We have run the numerical evolution of the scale factor with only the *classical* equation for η_{eff} with the proper initial conditions and found that it coincides within our numerical error with the evolution obtained by the full system of equations in either the classical or quantum case.

This remarkable feature of the zero mode assembly of long-wavelength spinodally unstable modes is a consequence of the presence of the horizon. It also explains why despite the fact that asymptotically when the fluctuations sample the broken symmetry state, the equation of state is that of matter. Since the excitations in the broken symmetry state are massless Goldstone bosons one would expect radiation domination. However, the 'assembly' phenomenon, i.e. the redshifting of the wave vectors, makes these modes behave exactly like zero momentum modes that give an equation of state of matter domination (upon averaging over the small oscillations around the minimum).

V. MAKING SENSE OF 'SMALL FLUCTUATIONS':

Having recognized the effective classical variable that can be interpreted as the component of the field that drives the FRW background and rolls down the classical potential hill, we want to recognize unambiguously the small fluctuations. We have argued above that after horizon crossing, all of the mode functions evolve proportionally to the zero mode, and the question arises: which modes are assembled into the effective zero mode and which modes are treated as perturbations? In principle every $k \neq 0$ mode provides some spatial inhomogeneity, and assembling these into an effective homogeneous zero mode seems in principle to do away with the very inhomogeneities that one wants to study. However, scales of cosmological importance today have first crossed the horizon during the last 60 or so e-folds of inflation. Recently Grishchuk [24] has argued that the sensitivity of the measurements of $\Delta T/T$ probe inhomogeneities on scales ≈ 500 times the size of the present horizon. Therefore scales that are larger than these and that have first crossed the horizon much earlier than the last 60 e-folds of inflation are unobservable today and can be treated as an effective homogeneous component, whereas the scales that can be probed experimentally via the CMB inhomogeneities today must be treated separately as part of the inhomogeneous perturbations of the CMB.

Thus a consistent description of the dynamics in terms of an effective zero mode plus 'small' quantum fluctuations can be given provided: a) the total number of e-folds $N_e \gg 60$, b) all the modes that have crossed the horizon *before* the last 60-65 e-folds are assembled into an effective *classical* zero mode via $\phi_{eff}(t) = [\phi_0^2(t) + \langle \psi^2(\vec{x}, t) \rangle_R]^{\frac{1}{2}}$, c) the modes that cross the horizon during the last 60–65 e-folds are accounted as 'small' perturbations. The reason for the requirement a) is that in the separation $\phi(\vec{x}, t) = \phi_{eff}(t) + \delta\phi(\vec{x}, t)$ one requires that $\delta\phi(\vec{x}, t)/\phi_{eff}(t) \ll 1$. As argued above after the modes cross the horizon, the ratio of amplitudes of the mode functions remains constant and given by $e^{(\nu - \frac{3}{2})\Delta N}$ with ΔN being the number of e-folds between the crossing of the smaller k and the crossing of the larger k. Then for $\delta\phi(\vec{x}, t)$ to be much smaller than the effective zero mode, it must be that the

Fourier components of $\delta\phi$ correspond to very large k's at the beginning of inflation, so that the effective zero mode can grow for a long time before the components of $\delta\phi$ begin to grow under the spinodal instabilities. In fact requirement a) is not very severe; in the figures (1-5) we have taken $h_0 = 1.5$ which is a very moderate value and yet for $\lambda = 10^{-12}$ the inflationary stage lasts for over 100 e-folds, and as argued above, the larger h_0 for fixed λ, the longer is the inflationary stage. Therefore under this set of conditions, the classical dynamics of the effective zero mode $\phi_{eff}(t)$ drives the FRW background, whereas the inhomogeneous fluctuations $\delta\phi(\vec{x}, t)$, which are made up of Fourier components with wavelengths that are much smaller than the horizon at the beginning of inflation and that cross the horizon during the last 60 e-folds, provide the inhomogeneities that seed density perturbations.

VI. SCALAR METRIC PERTURBATIONS:

Having identified the effective zero mode and the 'small perturbations', we are now in position to provide an estimate for the amplitude and spectrum of scalar metric perturbations. We use the clear formulation by Mukhanov, Feldman and Brandenberger [25] in terms of gauge invariant variables. In particular we focus on the dynamics of the Bardeen potential [26], which in longitudinal gauge is identified with the Newtonian potential. The equation of motion for the Fourier components (in terms of comoving wavevectors) for this variable in terms of the effective zero mode is [25]

$$\ddot{\Phi}_k + \left[H(t) - 2\frac{\ddot{\phi}_{eff}(t)}{\dot{\phi}_{eff}(t)} \right] \dot{\Phi}_k + \left[\frac{k^2}{a^2(t)} + 2 \left(\dot{H}(t) - H(t)\frac{\ddot{\phi}_{eff}(t)}{\dot{\phi}_{eff}(t)} \right) \right] \Phi_k = 0. \tag{6.1}$$

We are interested in determining the dynamics of Φ_k for those wavevectors that cross the horizon during the last 60 e-folds before the end of inflation. During the inflationary stage the numerical analysis suggests that to a very good approximation

$$H(t) \approx H_0 \; ; \; \phi_{eff}(t) = \phi_{eff}(0)e^{(\nu - \frac{3}{2})H_0 t}, \tag{6.2}$$

where H_0 is the value of the Hubble constant during inflation, leading to

$$\Phi_k(t) = e^{(\nu - 2)H_0 t} \left[a_k H_\beta^{(1)} \left(\frac{ke^{-H_0 t}}{H_0} \right) + b_k H_\beta^{(2)} \left(\frac{ke^{-H_0 t}}{H_0} \right) \right] \; ; \; \beta = \nu - 1 \; . \tag{6.3}$$

The coefficients a_k, b_k are determined by the initial conditions.

Since we are interested in the wavevectors that cross the horizon during the last 60 e-folds, the consistency for the zero mode assembly and the interpretation of 'small perturbations' requires that there must be many e-folds before the *last* 60. We are then considering wavevectors that were deep inside the horizon at the onset of inflation. Mukhanov et. al. [25] show that $\Phi_k(t)$ is related to the canonical 'velocity field' that determines scalar perturbations of the metric and which is quantized with Bunch-Davies initial conditions for the large k-mode functions. The relation between Φ_k and v and the initial conditions on v lead at once to a determination of the coefficients a_k and b_k for $k >> H_0$ [25]

$$a_k = -\frac{3}{2} \left[\frac{8\pi}{3M_{Pl}^2} \right] \dot{\phi}_{eff}(0) \sqrt{\frac{\pi}{2H_0}} \frac{1}{k} \; ; \; b_k = 0. \tag{6.4}$$

Thus we find that the amplitude of scalar metric perturbations after horizon crossing is given by

$$|\delta_k(t)| = k^{\frac{3}{2}}|\Phi_k(t)| \approx \frac{3}{2}\left[\frac{8\sqrt{\pi}}{3M_{Pl}^2}\right]\dot{\phi}_{eff}(0)\left(\frac{2H_0}{k}\right)^{\nu-\frac{3}{2}}e^{(2\nu-3)H_0 t}. \tag{6.5}$$

The power spectrum per logarithmic k interval is given by $|\delta_k(t)|^2$. The time dependence of $|\delta_k|$ displays the unstable growth associated with the spinodal instabilities of super-horizon modes and is a hallmark of the phase transition. This time dependence can be also understood from the constraint equation that relates the Bardeen potential to the gauge invariant field fluctuations [25], which in longitudinal gauge are identified with $\delta\phi(\vec{x},t)$. To obtain the amplitude and spectrum of density perturbations at *second* horizon crossing we use the conservation law associated with the gauge invariant variable [25]

$$\xi_k = \frac{\frac{2\dot{\Phi}_k}{3H} + \Phi_k}{1+p/\varepsilon} + \Phi_k \;\; ; \;\; \dot{\xi}_k = 0, \tag{6.6}$$

which is valid after horizon crossing of the mode with wavevector k. Using this conservation law, and the relation that during the inflationary stage $1+p/\varepsilon = 8\pi\dot{\phi}_{eff}^2/2M_{Pl}^2 H_0^2 \ll 1$, and assuming matter domination at second horizon crossing and $\dot{\Phi}_k(t_f) = 0$ [25], we find

$$|\delta_k(t_f)| = \frac{12\Gamma(\nu)\sqrt{\pi}}{5(\nu-\frac{3}{2})\mathcal{F}(H_0/m)}\left(\frac{2H_0}{k}\right)^{\nu-\frac{3}{2}}, \tag{6.7}$$

where $\mathcal{F}(H_0/m)$ determines the initial amplitude of the effective zero mode (4.18). We can now read the power spectrum per logarithmic k interval

$$\mathcal{P}_s(k) = |\delta_k|^2 \propto k^{-2(\nu-\frac{3}{2})}, \tag{6.8}$$

leading to the index for scalar density perturbations

$$n_s = 1 - 2(\nu - \frac{3}{2}). \tag{6.9}$$

We remark that we have not included the small corrections to the dynamics of the effective zero mode and the scale factor arising from the non-linearities. These are expected to lead to perturbatively small $\mathcal{O}(\lambda)$ corrections to the index (6.9). The spectrum given by (6.7) is similar to that obtained in references [2,10] although the amplitude differs from that obtained there. We emphasize an important feature of the spectrum: it has more power at *long wavelengths* because $\nu - 3/2 > 0$. This is recognized to be a consequence of the spinodal instabilities that result in the growth of long wavelength modes and therefore in more power for these modes. This seems to be a robust prediction of new inflationary scenarios in which the potential has negative second derivative in the region of field space that produces inflation.

It is at this stage that we recognize the consistency of our approach for separating the composite effective zero mode from the small fluctuations. We have argued above that many more than 60 e-folds are required for consistency, and that the 'small fluctuations' correspond

to those modes that cross the horizon during the last 60 e-folds of the inflationary stage. For these modes $H_0/k = e^{-H_0 t^*(k)}$ where $t^*(k)$ is the time of horizon crossing of the mode with wavevector k since the beginning of inflation. The scale that corresponds to the Hubble radius today $\lambda_0 = 2\pi/k_0$ is the first to cross during the last 60 or so e-folds before the end of inflation. Smaller scales today will correspond to $k > k_0$ at the onset of inflation since they will cross the first horizon later and therefore will reenter earlier. The bound on $|\delta_{k_0}| \propto \Delta T/T \leq 10^{-5}$ on these scales provides a lower bound on the number of e-folds required for these type of models to be consistent:

$$N_e > 60 + \frac{12}{\nu - \frac{3}{2}} - \frac{\ln(\nu - \frac{3}{2})}{\nu - \frac{3}{2}}, \tag{6.10}$$

where we have written the total number of e-folds as $N_e = H_0 \, t^*(k_0) + 60$. This in turn can be translated into an upper bound on the coupling constant using the estimate given by eq.(4.12).

VII. CONTACT WITH THE RECONSTRUCTION PROGRAM:

The program of reconstruction of the inflationary potential seeks to establish a relationship between features of the inflationary scalar potential and the spectrum of scalar and tensor perturbations. This program, in combination with measurements of scalar and tensor components either from refined measurements of temperature inhomogeneities of the CMB or through galaxy correlation functions will then offer a glimpse of the possible realization of the inflation [27,28]. Such a reconstruction program is based on the slow roll approximation and the spectral index of scalar and tensor perturbations are obtained in a perturbative expansion in the slow roll parameters [27,28]

$$\epsilon(\phi) = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{\dot{\phi}^2}{2} + V(\phi)} \,, \tag{7.1}$$

$$\eta(\phi) = -\frac{\ddot{\phi}}{H\dot{\phi}} \,. \tag{7.2}$$

We can make contact with the reconstruction program by identifying ϕ above with our ϕ_{eff} after the first few e-folds of inflation needed to assemble the effective zero mode from the quantum fluctuations. We have numerically established that for the weak scalar coupling required for the consistency of these models, the cosmologically interesting scales cross the horizon during the epoch in which $H \approx H_0$; $\dot{\phi}_{eff} \approx (\nu - 3/2) \, H_0 \, \phi_{eff}$; $V \approx m_R^4/\lambda \gg \dot{\phi}_{eff}^2$. In this case we find

$$\eta(\phi_{eff}) = \nu - \frac{3}{2} \; ; \; \epsilon(\phi_{eff}) \approx \mathcal{O}(\lambda) \ll \eta(\phi_{eff}). \tag{7.3}$$

With these identifications, and in the notation of [27,28] the reconstruction program predicts the index for scalar density perturbations n_s given by

$$n_s - 1 = -2(\nu - \frac{3}{2}) + \mathcal{O}(\lambda), \tag{7.4}$$

which coincides with the index for the spectrum given by the power spectrum per logarithmic interval $|\delta_k|^2$ with $|\delta_k|$ given by eq.(6.7). We must note however that our treatment did not assume slow roll for which $(\nu - \frac{3}{2}) \ll 1$. Our self-consistent, non-perturbative study of the dynamics plus the underlying requirements for the identification of a composite operator acting as an effective zero mode, validates the reconstruction program in weakly coupled new inflationary models.

VIII. CONCLUSIONS:

We have studied the non-equilibrium dynamics of a new inflation scenario in a self-consistent, non-perturbative framework based on a large N expansion, including the dynamics of the scale factor and backreaction of quantum fluctuations. Quantum fluctuations associated with superhorizon modes grow exponentially as a result of the spinodal instabilities and contribute to the energy momentum tensor in such a way as to end inflation consistently.

Analytical and numerical estimates have been provided that establish the regime of validity of the classical approach. We find that these superhorizon modes re-assemble into an effective zero mode and unambiguously identify the composite operator that can be used as an effective expectation value of the inflaton field whose *classical* dynamics drives the evolution of the scale factor. This identification also provides the initial condition for this effective zero mode.

If the model allows many more than 60 e-folds we provide a criterion that allows the identification of the small perturbations that give rise to scalar metric (curvature) perturbations. We then use this criterion combined with the gauge invariant approach to obtain the spectrum for scalar perturbations. We find the index to be less than one, providing more power at long wavelength as a result of the spinodal instabilities. We argue that this 'red' spectrum is a robust feature of potentials that lead to spinodal instabilities in the region in field space associated with inflation. Finally we made contact with the reconstruction program and validated the results for these type of models based on the slow-roll assumption, despite the fact that our study does not involve such an approximation. A more detailed version of this article with a discussion of the issue of decoherence is forthcoming [29].

IX. ACKNOWLEDGEMENTS:

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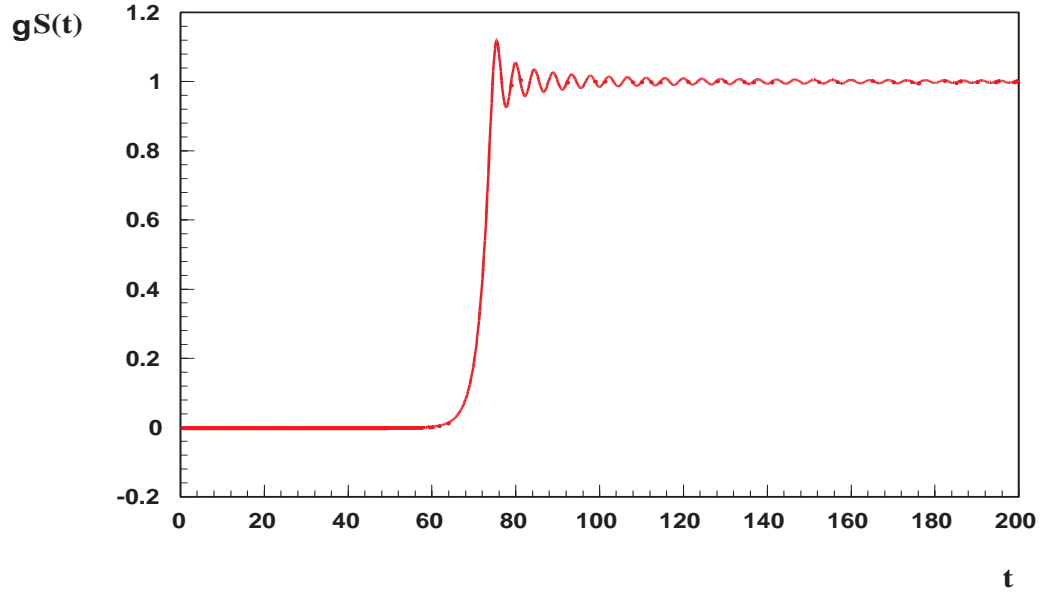[29] D. Boyanovsky, D. Cormier, H. J. de Vega, R. Holman and S. P. Kumar (in preparation).

FIGURES

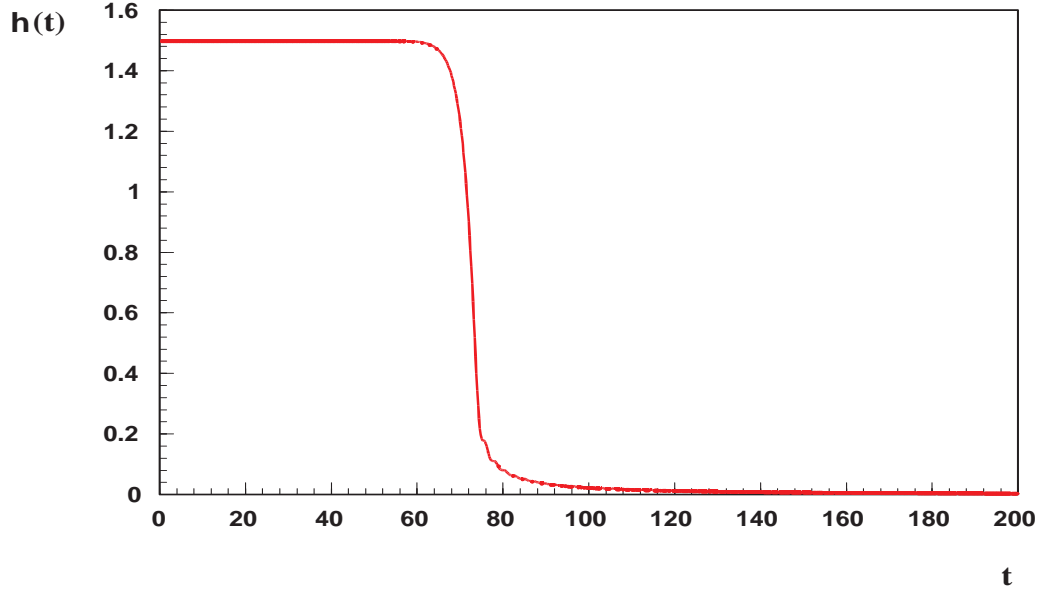FIG. 1. $g\Sigma$ vs. τ, for $\eta(0) = 0, \dot{\eta}(0) = 0, \lambda = 10^{-12}, h_0 = 1.5$

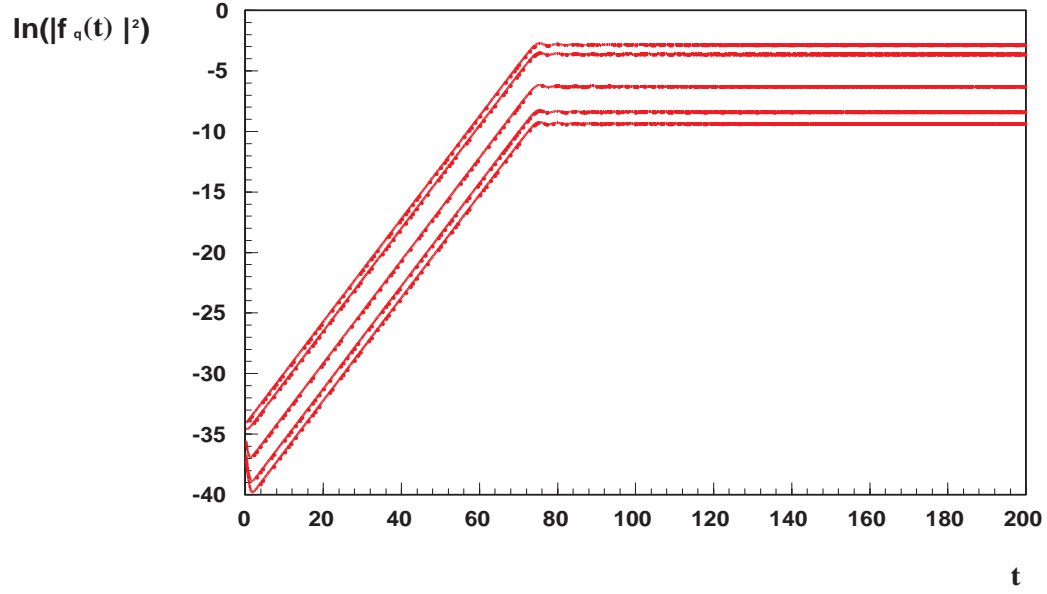FIG. 2. $H(\tau)$ vs. τ, for $\eta(0) = 0, \dot{\eta}(0) = 0, \lambda = 10^{-12}, h_0 = 1.5$

FIG. 3. $\ln(|f_q(\tau)|^2)$ vs. τ, for $\eta(0) = 0, \dot{\eta}(0) = 0, \lambda = 10^{-12}, h_0 = 1.5$ for $q = 0.5, 2.0, 5.0, 8.0, 10$ smaller q corresponds to larger values
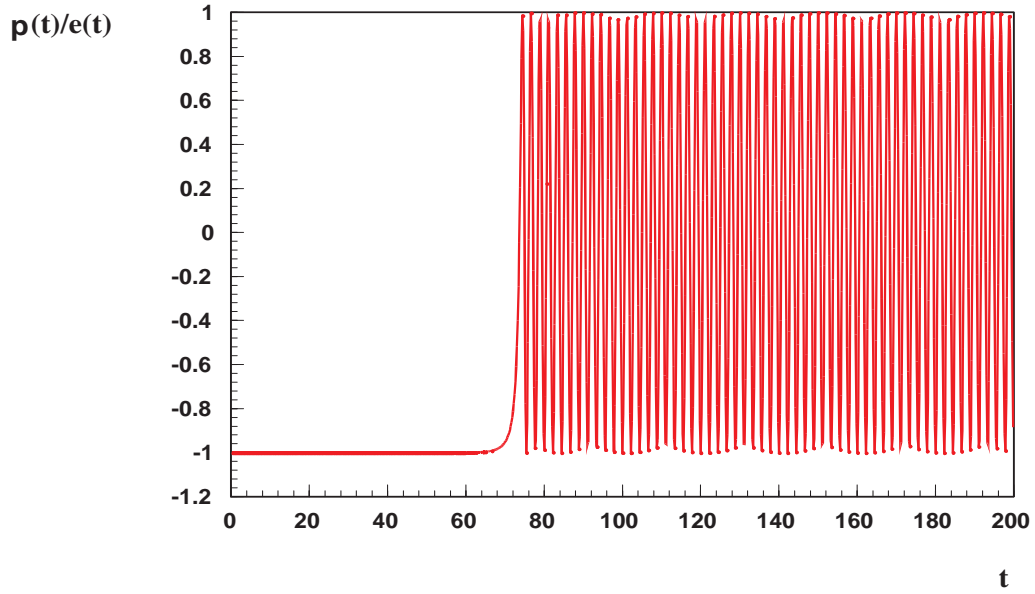

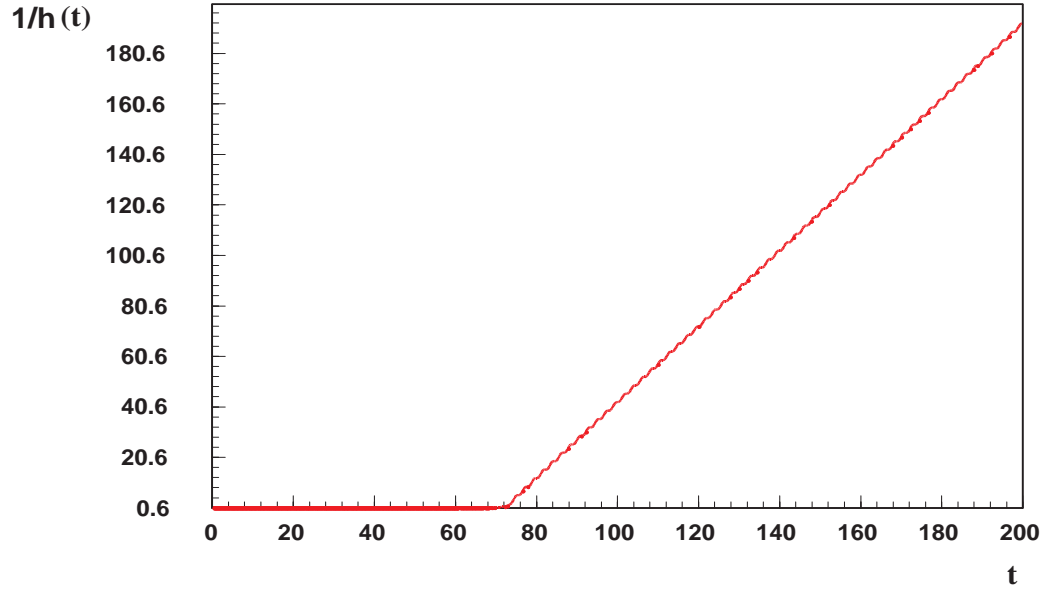FIG. 4. p/ε vs. τ, for $\eta(0) = 0, \dot{\eta}(0) = 0, \lambda = 10^{-12}, h_0 = 1.5$

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FIG. 5. $1/h(\tau)$ vs. τ, for $\eta(0) = 0, \dot{\eta}(0) = 0, \lambda = 10^{-12}, h_0 = 1.5$